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The theory of stability of spatially periodic parallel flows

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The stability of a parallel flow periodic in the direction normal to the stream is investigated theoretically. Critical Reynolds numbers are calculated for a general velocity profile including widely separated wakes. The critical mode of disturbance is found to have the same period as the basic flow. Growing modes with much larger periods exist, however, at slightly supercritical values of the Reynolds number. The analysis of various limiting cases explains the qualitative difference in the shape of the neutral curves depending on the period of the disturbance. In connection with the results obtained in this paper, the stability of non-parallel periodic flows is briefly discussed.

1. Introduction

The stability of parallel flows of a viscous incompressible fluid has been investigated for almost one century, and a great amount of data on stability characteristics has been obtained for all kinds of two-dimensional unidirectional flows. Most of the unbounded flows investigated so far, however, are those that are uniform at infinity; for example, wakes, jets, free shear layers and boundary-layer flows. Much less attention has been given to flows periodic in the direction normal to the stream. Recently, Beaumont (1981) investigated the linear stability of such a flow analytically in the inviscid limit and numerically at some finite Reynolds numbers. He studied, however, the viscous case in less detail than the inviscid case, and he did not give attention to velocity profiles other than the sinusoidal one. His paper was published just after the research reported in this paper was completed. So in this paper we investigate the linear stability problem of spatially periodic flows mainly for the viscous case, and calculate the critical Reynolds numbers for more-general velocity profiles.

The problem is also of much interest from a mathematical point of view. For a general velocity profile that is not necessarily periodic, the Orr–Sommerfeld equation allows a set of discrete eigenvalues and/or a set of continuous spectra. The eigenfunctions associated with these eigenvalues are supposed to form jointly a complete set. It has been proved that the eigenfunctions of the discrete eigenvalues form for themselves a complete set in the case of bounded flows such as plane Poiseuille flow or plane Couette flow (Lin 1961; DiPrima & Habetler 1969). So the continuous spectrum can appear only in unbounded flows, as demonstrated by Grosch

& Salwen (1978) for boundary-layer flow. But the continuous spectrum is damped and makes no essential contribution to the instability of the flow. Gustavsson (1979) and Salwen & Grosch (1981) showed that the eigenfunction for the continuous spectrum is obtainable by solving the Orr–Sommerfeld equation with a bounded boundary condition at infinity. In the case of the unbounded periodic flows to be investigated in this paper, the continuous eigenvalues play an absolutely important role because all the eigenvalues are found to be continuous.

The critical modes will be found in this paper to have the same period as the basic flow. At the same time, however, growing modes with a much longer period will be found at slightly supercritical values of the Reynolds number. This means that some larger-scale structures in space can develop in the unstable periodic flow. In fact, some flow patterns with double or triple period have been observed in the periodic wakes behind a row of cylinders (Matsui & Tamai 1974). Some kinds of grid turbulence contain spectral components of larger scale than the grid spacing. Quantitative accounts of these phenomena, however, are not available.

2. Formulation of the problem

Let $(U(y), 0, 0)$ be the velocity of a steady plane-parallel flow, taking the x-axis of a Cartesian coordinate system along the direction of the flow. Squire's theorem (Squire 1933) guarantees that we need consider only two-dimensional disturbances as long as only linearized theory is applied. The two-dimensional disturbance (u, v) can be expressed in terms of the stream function $\psi(x, y, t)$ as

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{2.1}$$

In particular, we consider here a harmonic component

$$\psi = \phi(y) \exp\left[i\alpha(x-ct)\right], \tag{2.2}$$

where α (> 0) represents the wavenumber in the x-direction, $c_r \equiv \mathrm{Re}\,(c)$ the phase velocity, and $\alpha c_i \equiv \alpha\,\mathrm{Im}\,(c)$ the amplification rate of the disturbance. Substituting (2.1) and (2.2) into the equation of motion and neglecting the nonlinear terms, we obtain the Orr–Sommerfeld equation:

$$(U-c)\,(D^2-\alpha^2)\,\phi - U''\phi = \frac{1}{i\alpha R}(D^2-\alpha^2)^2\,\phi, \quad D \equiv \frac{d}{dy}, \tag{2.3}$$

where the primes denote differentiation with respect to y, and R is the Reynolds number. Suppose that $U(y)$ is periodic and that the velocity and the space variables are made non-dimensional so as to satisfy the conditions $U_{\max} - U_{\min} = 1$ and $U(y+2\pi) = U(y)$. The velocity profile $U(y)$ is not any exact solution of the Navier–Stokes equation, but it can be considered as a simple model of some real periodic flows. The boundary condition for ϕ is given by

$$\phi(y) < \infty \quad \text{at} \quad y = \pm\infty, \tag{2.4}$$

as in the case of the boundary-layer flow.

The equation (2.3) forms a Floquet system, and Floquet theory permits us to assume each of the fundamental solutions consistent with (2.4) to have a form

$$\phi(y) = \exp\left[i\beta y\right] f(y), \tag{2.5}$$

where β is a *real* constant and $f(y+2\pi) = f(y)$. The parameter β specifies the period of the disturbances. The disturbance with $\beta = 1/n$, where n is a non-zero integer, has the period $2n\pi$, and the one with $\beta = 0$ has the same period 2π as the main flow, while an irrational value of β means an aperiodic disturbance. It should be remarked that this parameter does not appear in the original equation (2.3).

Substitution of (2.5) into (2.3) provides the equation for $f(y)$, namely,

$$(U-c)(\mathcal{D}^2-\alpha^2)f - U''f = \frac{1}{i\alpha R}(\mathcal{D}^2-\alpha^2)^2 f, \quad \mathcal{D} \equiv \frac{d}{dy}+i\beta, \tag{2.6}$$

which is just (2.3) with D replaced by $\mathcal{D}$. It is easily shown from (2.5) that the range of β can be reduced to the interval $(-\frac{1}{2},\frac{1}{2}]$. If $U(y)$ in (2.6) is an even function of y, this interval can be reduced further to the interval $[0,\frac{1}{2}]$, which is the case treated in this paper. By (2.5) the problem is reduced to one in the fundamental period $0 \leqslant y \leqslant 2\pi$.

Equation (2.6) contains a continuous parameter β. The eigenvalues of c may depend continuously on β. These eigenvalues therefore form the continuous spectrum stated before. Similarly the so-called neutral curve $R = R(\alpha,\beta)$ obtained by solving the equation $c_i = c_i(\alpha, R, \beta) = 0$ forms some two-dimensional area on the (R,α)-plane, which will be referred to as a 'neutral domain' in this paper. All the neutral curves should be found within the neutral domain. Also inside the domain some growing modes exist throughout, and the critical Reynolds number is determined by the neutral curve on one border of this domain. The eigenfunctions given by (2.5) do not vanish at infinity. On account of this behaviour, they are sometimes called *improper* eigenfunctions, in contrast with the ordinary localized eigenfunctions.

In the inviscid limit for bounded flows, several theorems have been established for the Rayleigh equation. Most of them can be extended to a periodic velocity profile, since the operator D in (2.3) and $\mathcal{D}$ in (2.6) are both antihermitian for real β and they are treated similarly in the partial integration.

3. Numerical calculation

For the numerical approach to (2.6) we have to give an appropriate form to the velocity distribution $U(y)$. Lorenz (1972), Green (1974) and Beaumont (1981) treated the sinusoidal profile $U(y) = \sin y$, which is a typical (and the simplest) form for a periodic velocity distribution. Another typical form is that of the wakes behind a row of cylinders with intervals much longer than the diameter. So, in order to investigate the different effects of various $U(y)$ profiles, we consider here a one-parameter family of velocity distributions which is simplified to a sinusoidal one in one limit and to periodic but sparsely spaced wakes in the other limit.

We take the following as such a basic velocity distribution:

$$\begin{aligned} U(y) &= C_1(\gamma) - C_2(\gamma) \sum_{n=-\infty}^{\infty} \exp\left[-\left\{\frac{\gamma}{2\pi}(y-2n\pi)\right\}^2\right] \\ &= C_1(\gamma) - C_2(\gamma)\frac{\pi^{\frac{1}{2}}}{\gamma}\vartheta_3\left(\tfrac{1}{2}y \,\middle|\, i\frac{\pi}{\gamma^2}\right) \\ &= C_1(\gamma) - C_2(\gamma)\left\{1 + 2\sum_{n=1}^{\infty} \exp\left[-\frac{\pi^2 n^2}{\gamma^2}\right]\cos ny\right\}, \end{aligned} \tag{3.1}$$

where γ is a positive parameter which varies from zero to infinity, C_1 and C_2 constants depending on γ, and ϑ_3 the Theta function of the third type (for its definition see

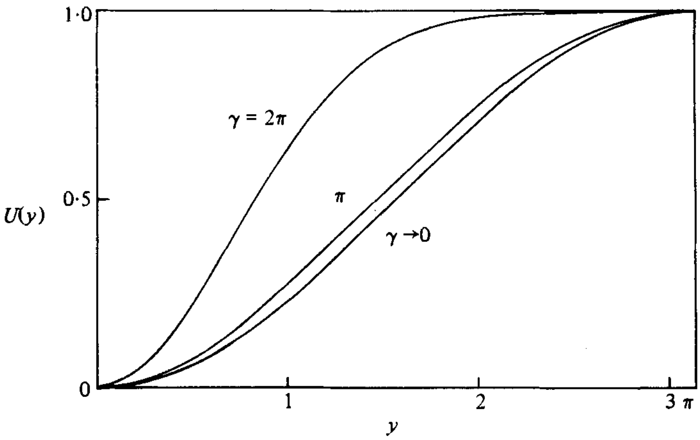

FIGURE 1. Velocity profiles of (3.1) for $\gamma = 2\pi$, π and $\gamma \to 0$.

Whittaker & Watson 1973, p. 464). The constant C_2 is fixed by the normalization condition $U_{\max} - U_{\min} = 1$ as

$$C_2(\gamma) = \frac{\dfrac{\gamma}{4\pi^{\frac{1}{2}}}}{\sum\limits_{n=0}^{\infty} \exp\left[-\dfrac{\pi^2}{\gamma^2}(2n+1)^2\right]}. \tag{3.2}$$

The other constant C_1 is an additive constant to $U(y)$. Arbitrary choice of it does not affect the stability. For analytical investigation it is convenient to choose C_1 equal to C_2 in order that

$$\int_0^{2\pi} U(y)\,dy = 0.$$

But in this section we choose C_1 such that $U_{\min} = 0$ for convenience in computation. It is of course easy to transform results for one case to the other.

The velocity profile (3.1) has the following asymptotic forms with respect to γ (figure 1):

$$U(y) \sim \begin{cases} 1 - \exp\left[-\left(\dfrac{\gamma}{2\pi}y\right)^2\right] \quad (\gamma \to \infty, \quad y \sim 0), & (3.3a) \\ \tfrac{1}{2}(1-\cos y) \quad (\gamma \to 0). & (3.3b) \end{cases}$$

Thus, as stated before, $U(y)$ is reduced to the velocity profile corresponding to sparsely spaced wakes as $\gamma \to \infty$, and to that of a sinusoidal flow as $\gamma \to 0$. In other words, (3.1) can express various periodic profiles extending from weak to strong interference of adjacent wakes.

To solve (2.6) numerically we made use of spectral method with Fourier series. We expanded $U(y)$ and $f(y)$ into Fourier series and then truncated them as follows:

$$U(y) = \sum_{n=-N}^{N} U_n e^{iny}, \quad f(y) = \sum_{n=-N}^{N} f_n e^{iny}. \tag{3.4}$$

Substituting (3.4) into (2.6), we obtained the eigenvalue problem of a finite-dimensional matrix:

$$[M_{ij}]^{\mathrm{t}}[f_{-N}, \ldots, f_N] = c\,{}^{\mathrm{t}}[f_{-N}, \ldots, f_N], \tag{3.5}$$

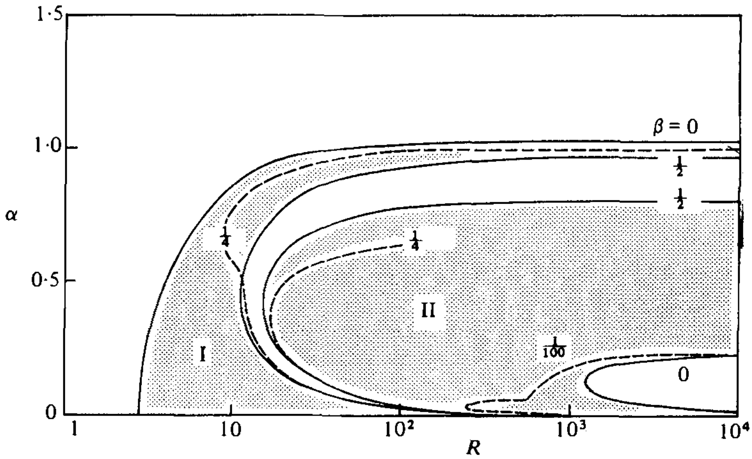

FIGURE 2. Neutral-stability curves for $\gamma = \pi$: (I) the first neutral domain; (II) the second neutral domain.

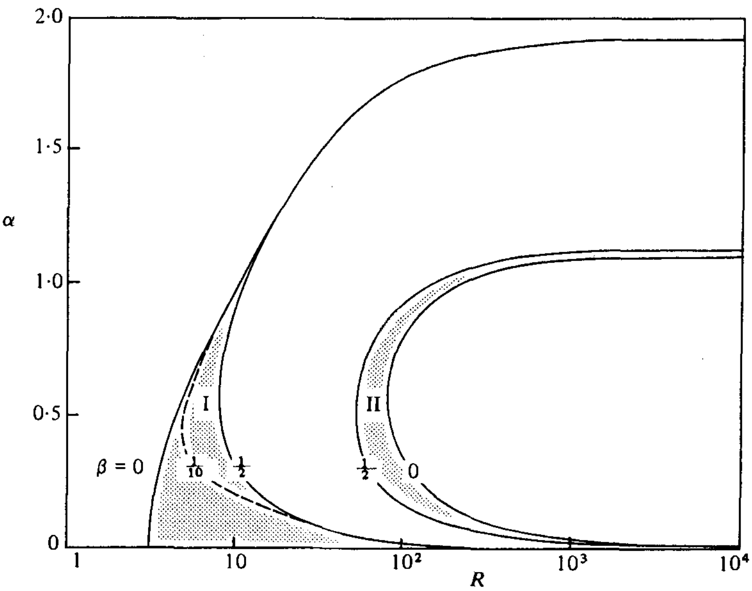

FIGURE 3. Neutral-stability curves for $\gamma = 2\pi$: (I) the first neutral domain; (II) the second neutral domain.

where
$$M_{N+1+m,N+1+n} = \frac{U_{m-n}}{A_m}\{A_n-(m-n)^2\}+\frac{A_m}{i\alpha R}\delta_{mn},$$
$$A_m = (m+\beta)^2+\alpha^2 \quad (m,\, n = -N, \ldots, N),$$
and $\delta_{mn} = 1$ if $m = n$, but is zero if $m \neq n$. We solved (3.5) by increasing N until convergence was plausibly obtained for the eigenvalues of c. In most cases, $N = 10, 15, 20$ were found to be sufficient to reduce the errors within one per cent.

The calculated neutral curves are shown in figure 2 for $\gamma = \pi$, and in figure 3 for $\gamma = 2\pi$. In each case, there are two separate domains in the (R, α)-plane. The left neutral domain in figure 2 is constructed by the neutral curves with all $\beta \in [0, \frac{1}{2}]$, where

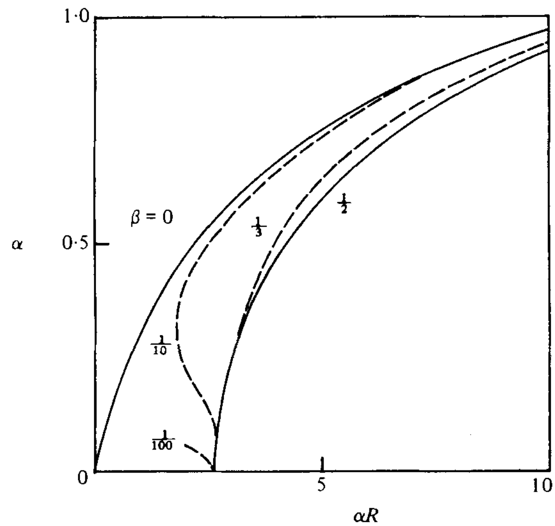

FIGURE 4. Neutral-stability curves for $\gamma = 2\pi$ in the first domain plotted on $(\alpha R, \alpha)$-plane.

the curve with $\beta = 0$ is on the left border of the domain, while that with $\beta = \frac{1}{2}$ is near the right border. The right neutral domain is also constructed by the neutral curves with all $\beta \in [0, \frac{1}{2}]$, the order of which is, however, converse to that of the left one. Let us call these two domains the first and second domains respectively. Both neutral domains become rapidly larger as γ varies from 2π to π, that is, as the effects of neighbouring wakes are intensified. Growing modes are to be found everywhere to the right of the neutral curves. The neutral curves of the second domain indicate a distinct family of modes which are growing to the right of these curves along with the other modes with respect to the first domain.

On each of the neutral curves in both domains, α approaches some finite non-zero value at one end but zero on the other end. The shape is similar to that of the neutral curve for the single wake (Taneda 1963). Beaumont (1981) showed, in the case of $U(y) = \sin y$, that the wavenumber band of the unstable mode is separated into two parts with $c_r = 0$ and $c_r \neq 0$ for some fixed values of β at $R = 14$, 20 and 40. For general periodic flows, the band may separate not into the simple pair stated above, but into some more complicated parts. The 'forehead' shape of the neutral curves in our calculation, for example the curve for $\gamma = \pi$ and $\beta = \frac{1}{4}$ in the first domain, may be a reflection of this separation, but we do not pursue this problem in detail here. A brief comment will be added in §4.

In every case calculated in the present investigation, the critical Reynolds number $R_c = \min_{\alpha,\beta} R(\alpha, \beta)$ is given for $\alpha = \beta = 0$. This result is consistent with Green's comment that the most unstable mode would have the same period as the main flow (Green 1974). He showed that, in the case of $U(y) = \sin y$, $R(\alpha, \beta)$ approaches a non-zero and finite value as $\alpha \to 0$ on the neutral curve for the antisymmetric disturbance ($\phi(-y) = \phi(y)$) with $\beta = 0$. This is also true for our velocity profile (3.1), and it will be shown in §4 that this holds for every periodic velocity profile. An integral formula for evaluating $R(0, 0)$ exactly will be given there which yields results that coincide exactly with those obtained by the numerical calculation.

In the case of parallel flows with rigid boundaries, R goes to infinity as $\alpha \to 0$ on a lower branch of the neutral curve (Synge 1938), and to zero or infinity in the case of unbounded free flows that become uniform as $y \to \pm\infty$ (Tatsumi & Gotoh 1960; Tatsumi & Kakutani 1958). The lower branch of the neutral-stability curve, which gives a non-zero and finite Reynolds number in the long-wave limit, is therefore a remarkable characteristic of the periodic velocity profile.

In contrast, in the cases where $\beta \neq 0$, the asymptotes of the lower branch of the neutral curves are $R \to \infty$ as $\alpha \to 0$. This means that, in the long-wave limit, the unstable modes are only those with the same period as the main flow, which will be shown analytically in the following sections for an arbitrary periodic but even velocity profile. Figure 4, in which the curves are replotted on the $(\alpha R, \alpha)$-plane, shows that, on a curve for fixed non-zero $\beta, \alpha R$ approaches some constant independent of β, at least within the numerical error, as $\alpha \to 0$.

Regarding the spatial symmetry of the eigenfunction, we found that the eigenfunction with $\beta = 0$ in the first domain is even, and the one with $\beta = 0$ in the second domain is odd. Except for these cases, no symmetry could be found.

4. Critical Reynolds number

In this section and §5 we proceed with the analytical approach to the problem, the results of which will justify the numerical results obtained in §3. The long wave is treated in this section on the neutral curve with $\beta = 0$ in the first domain. The asymptotic behaviour of the neutral curves will be treated in §5. In these sections some types of integration appear frequently, and so we use a simplified notation defined as follows for arbitrary integrable functions $f(y)$ and $g(y)$:

$$\oint f \equiv \int_0^{2\pi} f(y)\,dy, \quad \oint fg = \int_0^{2\pi} f(y)\,g(y)\,dy, \tag{4.1 a, b}$$

$$\langle f\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(y)\,dy \quad \left(=\frac{1}{2\pi}\oint f\right), \tag{4.1 c}$$

$$\bar{f}(y) \equiv \int_0^{y} f(y)\,dy - \left\langle \int_0^{y} f(y)\,dy \right\rangle, \tag{4.1 d}$$

the first being the integral over a single period, the second the mean over a single period, and the third the indefinite integral in which the integration constant is determined so that the mean of the integrated function vanishes. It is easily seen that they have the following properties:

$$\frac{d}{dy}\bar{f}(y) = f(y), \tag{4.2a}$$

$$\bar{f}(y) \text{ is periodic} \quad \text{if} \quad \oint f = 0, \tag{4.2b}$$

$$\oint f\bar{g} = -\oint \bar{f}g \quad \text{if} \quad \oint f = \oint g = 0. \tag{4.2c}$$

Lastly, $U(y)$ denotes an arbitrary periodic velocity profile if not specified, the normalization of which is made by the condition $\oint U = 0$, as stated in §3. Another normalization is the same as before.

We expand the quantities in (2.6) into power series in the wavenumber α:

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots, \tag{4.3a}$$

$$R = R_0 + R_1\epsilon + R_2\epsilon^2 + \dots, \tag{4.3b}$$

$$c = c_0 + c_1\epsilon + c_2\epsilon^2 + \dots, \tag{4.3c}$$

where $\epsilon \equiv i\alpha$, all the f_n are periodic functions with period 2π, and the coefficients R_n and c_n are real for even n and pure-imaginary for odd n. Substituting (4.3) into the following form of (2.6),

$$f^{\mathrm{iv}} = \epsilon R\{(U-c)f'' - U''f\} - 2\epsilon^2 f'' + \epsilon^3 R(U-c)f - \epsilon^4 f, \tag{4.4}$$

we get successive equations for f_n, namely,

$$\begin{aligned} f_n^{\mathrm{iv}} = &\sum_{j+k=n-1} R_j\left\{Uf_k'' - U''f_k - \sum_{l+m=k} c_l f_m''\right\} \\ &-2f_{n-2}'' + \sum_{j+k=n-3} R_j\left\{Uf_k - \sum_{l+m=k} c_l f_m\right\} - f_{n-4} \quad (n = 0, 1, 2, \dots), \end{aligned} \tag{4.5}$$

where the f_k, R_k and c_k with negative k should be put equal to zero. Some ambiguities have still been left in the f_n as to the components parallel to f_0, for removal of which we can assume $\oint f_0 f_n = 0$ for $n \neq 0$.

Since the f_n are periodic, the integral of the right-hand side of (4.5) over one period should vanish, i.e.

$$\sum_{j+k=n-3} R_j\left[\oint Uf_k - c_k\oint 1\right] = \delta_{n4}\oint 1, \tag{4.6}$$

where use has been made of the fact that $f_0 = 1$ from (4.5) and the normalization of f_0. It can be proved that (4.6) is a sufficient condition for the f_n to be periodic. Thus (4.6) is the solvability condition for (4.5). If f_0 is noted to be the solution of the adjoint homogeneous equation of (4.5), the solvability condition (4.6) is of course obtained by the familiar procedure.

By solving (4.5) for $n = 1$, we have

$$f_1 = -R_0\overline{\overline{U}}, \tag{4.7}$$

where the bar operation is defined in (4.1d). Thus f_1 is real and even if $U(y)$ is also even. Putting $n = 3$ in (4.6), we have $R_0 c_0 = 0$. Equation (4.6) becomes for $n = 4$

$$R_0\left[\oint Uf_1 - c_1\oint 1\right] - R_1 c_0\oint 1 = \oint 1. \tag{4.8}$$

Then $c_0 = c_1 = 0$, and R_0 is obtained from (4.2), (4.7) and (4.8) as follows:

$$\frac{1}{R_0^2} = \frac{1}{R_0}\frac{\oint Uf_1}{\oint 1} = -\frac{\oint U\overline{\overline{U}}}{\oint 1} = \langle(\overline{U})^2\rangle. \tag{4.9}$$

This is the formula for the limiting Reynolds number R_0 as $\alpha \to 0$ on the neutral curve with $\beta = 0$ in the first domain.

By the method of mathematical induction, it can be proved for all integers n that

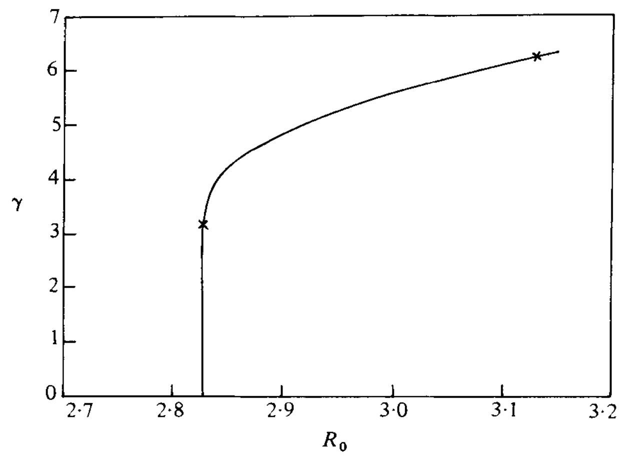

FIGURE 5. R_0 for the velocity profiles (3.1). The two crosses show the values $(R_0, \gamma) = (2{\cdot}828, \pi)$ and $(3{\cdot}129, 2\pi)$ obtained by the numerical calculation in §3.

$c_{2n+1} = R_{2n+1} = 0$ and that f_n is real and even if $U(y)$ is. Also, the f_n, c_{2n} and R_{2n} can be obtained successively from (4.6) and (4.8), for example

$$f_2 = R_0^2(U''\overline{\overline{U}} - U^2), \tag{4.10a}$$

$$c_2 = R_0^2\langle \overline{\overline{U}}(\overline{U''\overline{\overline{U}} - U^2})\rangle, \tag{4.10b}$$

$$R_2 = -\tfrac{1}{2}R_0^2[3R_0\langle(\overline{\overline{U}})^2\rangle + \langle\overline{\overline{U}}(\overline{Uf_2' - U'f_2})\rangle]. \tag{4.10c}$$

In §3, the neutral mode, obtained analytically here, was the critical mode, and R_0, obtained in (4.9), was the critical Reynolds number. This is also in fact for the sinusoidal velocity profile considered by Green (1974) and others. For a general flow, however, at least the second proposition does not always seem to be true, because the expression for R_2 in (4.10) indicates the possibility that R_2 can take a positive value for a particular velocity profile. This conjecture will be supported by the following discussion.

Using the formula (4.9) for the velocity profile (3.1), we obtain R_0 explicitly:

$$R_0 = \sqrt{8}\,\frac{\sum_{n=0}^{\infty} \exp\left[-\dfrac{\pi^2}{\gamma^2}(2n+1)^2\right]}{\left[\sum_{n=1}^{\infty} \dfrac{1}{n^2}\exp\left(-\dfrac{2\pi^2}{\gamma^2}n^2\right)\right]^{\frac{1}{2}}}, \tag{4.11}$$

which is shown graphically in figure 5. Perfect agreement is shown between the analytical and the numerical results. Equation (4.11) is reduced asymptotically to

$$R_0 \sim \begin{cases} \sqrt{8}\left(1 - \tfrac{1}{8}\exp\left(-\dfrac{6\pi^2}{\gamma^2}\right) + \exp\left(-\dfrac{8\pi^2}{\gamma^2}\right) + \ldots\right) & (\gamma \to 0), \quad (4.12a) \\[2ex] \dfrac{3\gamma^2}{\pi^3} & (\gamma \to \infty). \quad (4.12b) \end{cases}$$

Thus, in the limit of small values for γ, R_0 approaches $\sqrt{8}$, the value for the sinusoidal velocity profile obtained by Green (1974) by making use of a special relation between the Fourier components valid only for the sinusoidal flow. (This value is $\sqrt{2}$ in his paper owing to different scaling.) In the limit of large value of γ, R_0 diverges as the square of γ. In this case the main flow becomes similar to a single wake. Then, for a sufficiently large value of γ, it is reasonable that the critical Reynolds number is approximately that for the single wake. In this approximation, rescaling shows that the critical Reynolds number depends linearly on γ when γ is large. Thus R_0 goes to infinity faster than the critical Reynolds number as $\gamma \to \infty$. This means that, for a large value of γ, R_0 can never be the critical Reynolds number, and in this case R_2 in (4.3) must have a positive value as conjectured.

Finally, it should be remarked that the sinusoidal velocity profile may be typical but is rather special for investigation of the stability of periodic flows, because in such a case the factor $U''\overline{\overline{U}} - U^2$, which appears in each in (4.10*a*–*c*), vanishes exactly and then f_2 and c_2 vanish. This is presumably the reason why the condition $c_r = 0$ can distinguish the unstable modes for sinusoidal flow (Beaumont 1981). In the general case, these modes may be separated into some more complicated sets as stated in §3.

5. Asymptotic behaviour of the neutral curves

The neutral curves presented in §3 exhibit remarkable distribution of the asymptotes. An important part of the dependence of the distribution upon β can be made clear by the following analysis.

First, we consider the growth rates of the long-wave disturbances with small values of β. Define a small parameter $\epsilon \equiv \alpha R$ and put

$$\alpha = \alpha_1 \epsilon, \quad \beta = \beta_1 \epsilon, \tag{5.1}$$

where new parameters α_1 and β_1 are assumed independent of ϵ. Substituting (5.1) and the power-series expansion

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots, \tag{5.2a}$$

$$c = c_0 + \epsilon c_1 + \epsilon^2 c_2 + \dots \tag{5.2b}$$

into (2.6), we get the equations for f_n, namely,

$$\begin{aligned} f_n^{\mathrm{iv}} = & -i \sum_{m=1}^{n} c_{m-1} f''_{n-m} + 2\beta_1 \sum_{m=1}^{n-1} c_{m-1} f'_{n-1-m} \\ & + i(U f''_{n-1} - U'' f_{n-1}) - 4i\beta_1 f'''_{n-1} \\ & + i(\alpha_1^2 + \beta_1^2) \sum_{m=1}^{n-2} c_{m-1} f_{n-2-m} + 2(\alpha_1^2 + 3\beta_1^2) f''_{n-2} \\ & - 2\beta_1 U f'_{n-2} - i(\alpha_1^2 + \beta_1^2)\, U f_{n-3} \\ & + 4i\beta_1(\alpha_1^2 + \beta_1^2) f'_{n-3} - (\alpha_1^2 + \beta_1^2)^2 f_{n-4}. \end{aligned} \tag{5.3}$$

As in §4, the f_n are assumed to satisfy the orthogonality condition $\oint f_0 f_n = \oint \delta_{n0}$.

In the case of both $U(y)$ and $f(y)$ being even when $\beta = 0$, as analysed in §4, we can consistently put each of the f_n in the form

$$f_n = g_n + \beta_1 h_n, \tag{5.4}$$

where the g_n are even and the h_n odd. Substitution of (5.4) into (5.3) and separation of the resultant equation into even and odd parts provide the equations for g_n and h_n. The solvability conditions of these equations can easily be obtained in the same manner as before; from the equation for g_n

$$\frac{2i\beta_1^2}{\alpha_1^2+\beta_1^2}\oint Uh'_{n-2}-\oint Ug_{n-3}+c_{n-3}\oint 1+i(\alpha_1^2+\beta_1^2)\oint \delta_{n4}=0, \tag{5.5}$$

while the other condition is automatically satisfied.

Now it can be seen essential to treat the following two cases separately:

$$\text{case (i)}\quad \beta_1\to 0 \quad\text{then}\quad \alpha_1\to 0,$$

$$\text{case (ii)}\quad \alpha_1\to 0 \quad\text{then}\quad \beta_1\to 0,$$

because the solvability condition (5.5) generates the following different condition in each case:

$$\text{case (i)}\quad c_{n-3}\oint 1-\oint Ug_{n-3}=0, \tag{5.6a}$$

$$\text{case (ii)}\quad c_{n-3}\oint 1-\oint Ug_{n-3}+2i\oint Uh'_{n-2}=0, \tag{5.6b}$$

while in both cases the equations for g_n and h_n become

$$g_n^{\text{iv}}=-i\sum_{m=1}^{n} c_{m-1}g''_{n-m}+i(Ug''_{n-1}-U''g_{n-1}), \tag{5.7}$$

$$h_n^{\text{iv}}=-i\sum_{m=1}^{n} c_{m-1}h''_{n-m}+i(Uh''_{n-1}-U''h_{n-1}) \\ +2\sum_{m=1}^{n-1} c_{m-1}g'_{n-1-m}-4ig'''_{n-1}-2Ug'_{n-2}. \tag{5.8}$$

This non-uniformity just corresponds to the two distinct branches for the neutral curves shown in figure 4.

Solving (5.7) and (5.8) for $n=0$ and 1, we obtain

$$g_0=1,\quad g_1=-i\overline{\overline{U}},\quad h_0=h_1=0. \tag{5.9}$$

Substitution of (5.9) into (5.6a, b) with $n=3$ leads to $c_0=0$ in both cases, and then the solutions of (5.7) and (5.8) with $n=2$ are obtained as

$$g_2=\overline{\overline{\overline{\overline{(U^2-U''\overline{\overline{\overline{U}}})}}}},\quad h_2=-4\overline{\overline{\overline{U}}}. \tag{5.10}$$

From (5.9), (5.10) and (5.6a, b) with $n=4$, we obtain

$$\text{case (i)}\quad c_1=i\langle(\overline{U})^2\rangle=\frac{i}{R_0^2}, \tag{5.11a}$$

$$\text{case (ii)}\quad c_1=-7i\langle(\overline{U})^2\rangle=-\frac{7}{R_0^2}i. \tag{5.11b}$$

Thus the mode in case (i) is a growing mode, while that in case (ii) is a decaying mode. This agrees with the conclusion of the numerical calculation that the growing mode is only one with the same period as the main flow in the long-wave limit for fixed R (figure 4). If we do not take the limit $\alpha_1\to 0$ while putting β_1 zero in case (i) then we can get, after a similar calculation to that above,

$$c_1=i\left[\langle(\overline{U})^2\rangle-\frac{1}{R^2}\right]. \tag{5.12}$$

Thus the expression (4.9) is again obtained on the neutral curve where c_1 is real.

Next we consider the upper branch of the neutral curves. It seems appropriate to remark here that when the value of c can be fixed so as to make $U''/(U-c)$ regular, the inviscid limit of (2.6) is just the Schrödinger equation with the periodic potential $U''/(U-c)$. In the latter equation, β represents the wavenumber of the wave function and $-\alpha^2$ the energy eigenvalue, and the range of $-\alpha^2$ is called the energy band, which is a fundamental concept in solid-state physics. In the band theory, it has been proved that $\partial\alpha/\partial\beta \neq 0$ unless $\beta = 0$ or $\frac{1}{2}$ (Jones 1975). Thus, if the negative-energy eigenvalues are obtainable for all β between 0 and $\frac{1}{2}$, then each neutral domain should be bordered by the neutral curves with $\beta = 0$ or $\frac{1}{2}$, as shown by our numerical calculation.

When $\beta = 0$, Rayleigh's inflexion-point theorem and the converse theorem can be extended to the velocity profile $U(y)$ for which there exists a value of c such that $U''/(U-c)$ has no singularity and keeps its sign in the fundamental period of y. The extension assures at least a real periodic solution of the Rayleigh equation, $f_1(y)$ say, together with the eigenvalue of α, α_s say, and c determined by the above condition.

For small values of β, the perturbation calculus provides the expression of the eigenvalue of α in the form:

$$\alpha = \alpha_s + \frac{A}{2\alpha_s}\beta^2, \tag{5.13}$$

where

$$A = -\frac{4\pi^2 f_1(0)}{\{f_2(2\pi)-f_2(0)\}\oint f_1^{\,2}}, \tag{5.14}$$

and $f_2(y)$ is the solution of the Rayleigh equation linearly independent of $f_1(y)$. The functions f_1 and f_2 have been normalized so as to make their Wronskian equal to unity.

For the largest eigenvalue of α_s it has been known that the eigenfunction $f_1(y)$ does not vanish in the interval $[0, 2\pi]$ (see e.g. Ince 1956), which enables us to have

$$f_2(2\pi)-f_2(0) = f_1(0)\oint f_1^{-2}. \tag{5.15}$$

Thus A defined by (5.14) is found to be negative-definite, and in the first neutral domain α decreases as β increases from zero. The eigenfunction for the second-largest eigenvalue of α_s, on the other hand, vanishes at $y = 0$ and π. It makes (5.15) inapplicable, and we have to go back to (5.14) for the evaluation of A.

It is well known that the Rayleigh equation is rather easy to solve numerically. For the eigenvalues, however, it is easier to use an approximation on the basis of the formal analogy between the inviscid equation and the Schrödinger equation. When c is fixed so as to make $U''/(U-c)$ a regular function for any symmetric $U(y)$, it can be expanded into an even power series in y:

$$\frac{U''}{U-c} = B_0 + B_1 y^2 + B_2 y^4 + \dots. \tag{5.16}$$

Using the simplest approximation in which the first two terms of the expansion (5.16) are taken into consideration, we can easily obtain the solution of the eigenvalue problem. Because α^2 must be non-negative, only a finite number of the solutions are physically permissible. The improved solution can be obtained by taking the quartic term in (5.16) into account as a perturbation (see e.g. Feynman 1972), namely,

$$\alpha_n^2 = -B_0 - (2n+1)B_1^{\frac{1}{2}} - \frac{3B_2}{4B_1}(2n^2+2n+1) \quad (n = 0, 1). \tag{5.17}$$

Only two real eigenvalues are obtained here, consistent with the numerical results in §3. The values $\alpha_0 = 1{\cdot}916$ and $\alpha_1 = 1{\cdot}039$ thus obtained in the case of $\gamma = 2\pi$ show fairly good agreements with the results in §3.

It should be remarked that the approximation is appropriate when the profile of the main flow is a superposition of widely separated flows, because in this case the eigenfunction takes significant values only in the region where the approximation (5.16) is valid. Even in such a case, however, the precise evaluation of A cannot be done by the present approximation, because inaccurate small values of $f_1(y)$ at points not close to $y = 0$ make a significant contribution to $f_2(2\pi)$. Nevertheless, it is almost evident that the magnitude of A will be very small, which explains qualitatively the drastic convergence of the first domain in the inviscid limit in figure 2. For the main flow of periodic broad wakes, we should consider the problem using more-refined techniques as in the band theory in solid-state physics. However, in such a case the direct numerical calculation may be most useful.

6. Discussion

In this paper we have examined the stability of a spatially periodic parallel flow. No discrete eigenvalues exist in this problem, and all the eigenvalues are continuous and all the eigenfunctions are *improper* in the sense that they do not vanish at infinity.

Existence of the continuous eigenvalue has been known in the problem of a boundary-layer flow on the grounds that only a finite number of discrete eigenvalues are allowed to exist, so that an arbitrary disturbance needs improper eigenfunctions in its evolution. In boundary-layer flow, however, all the continuous modes have been found to be decaying, and so it is the Tollmien–Schlichting wave which plays the essential role in the instability of the flow. This conclusion is valid, as was proved by Grosch & Salwen (1978), for all unbounded flow that is uniform at infinity. In the case of periodic flows, on the other hand, it is not applicable, that is, only continuous eigenvalues exist, for which growing modes are found. This is a most important characteristic of unbounded periodic flows, and leads to the continuously distributed neutral curves defining the neutral domains.

An arbitrary disturbance in the periodic flow must be expressed by only improper eigenfunctions and they contain a factor exp $(i\beta y)$, so the disturbance propagates obliquely across the undisturbed stream in the (x, y)-plane. Since β is a continuous function of c, the disturbance initially in the form of a wave packet will propagate with some angular distribution of intensity. We do not know so far whether this phenomenon has been or can be observed in a suitable experiment.

The critical Reynolds number R_c in our calculation is always associated with the modes of the same period as the main flow. However, at the Reynolds number in an arbitrary upper neighbourhood of R_c, the growing modes of much longer periods have been found to exist. So when the flow becomes unstable, these modes will grow almost simultaneously, and then the flow may have a larger-scale structure than the original flow.

The critical Reynolds number obtained in §2 is not large enough for the fundamental assumption of parallel flow to be valid exactly, or even approximately. However, the existence of the continuous spectrum and the appearance of a larger-scale structure are postulated for an exact non-parallel periodic solution $\mathbf{U}(\mathbf{x})$ of the Navier– Stokes equation as follows. Suppose that $\mathbf{U}(\mathbf{x})$ has the periods $\mathbf{a}$ and $\mathbf{b}$ as $\mathbf{U}(\mathbf{x}) = \mathbf{U}(\mathbf{x}+\mathbf{a}) = \mathbf{U}(\mathbf{x}+\mathbf{b})$. This type of periodic flow is realized, for example, when a grid is set transversely to a uniform flow. The linearized equations for the disturbance $\mathbf{u}(\mathbf{x}, t)$ in such a non-parallel periodic flow consist of a set of partial differential equations involving some periodic functions with the same periods as $\mathbf{U}(\mathbf{x})$. This set of equations is invariant to the translational group generated by **a** and **b**. The representation theory of groups (Tinkham 1974) ensures that the solutions of such equations, which remain finite at infinity, have the form $\mathbf{u}(\mathbf{x}, t) = \exp(i\boldsymbol{\beta}\mathbf{x} + \sigma t)\,\mathbf{f}(\mathbf{x})$, where $\mathbf{f}(\mathbf{x})$ has the same period as the main flow, $\boldsymbol{\beta}$ is an arbitrary real constant vector and σ is the growth rate of the disturbance. The argument so far is rigorous and the result is a generalization of Floquet's theorem. On account of the continuous parameter $\boldsymbol{\beta}$ all spectra are continuous, as in the present investigation. As a consequence we obtain the conclusion that if the flow could be in a supercritical state then some larger-scale structure than the main flow would appear in it. The eigenvalue problem to prove the existence of the critical Reynolds number in this case would surely be much more difficult than in the present investigation.

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